Ph634: Advanced Quantum Mechanics / Quantum Field Theory
Problem Set 2: Dirac Fields

(1) Consider the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

(a) Show that γ^5 anticommutes with each of the γ^μ matrices, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

(b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

(c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$.

(Sign convention: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$.)

(d) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$ and γ^5.

(2) Under continuous Lorentz symmetries, Dirac spinor fields $\Psi(x)$ transform according to $\Psi'(x') = M(L)\Psi(x = L^{-1}x')$ where $M(L = e^\theta)) = \exp(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta})$. Consider the transformation rules for the independent bilinears $\overline{\Psi}\Gamma\Psi$, namely

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi \quad \text{and} \quad P = \overline{\Psi}\gamma^5\Psi.$$

Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors and the $T^{\mu\nu}$ as an antisymmetric tensor.

Under the *parity* symmetry $\mathcal{P} : (\vec{x}, t) \mapsto (-\vec{x}, t)$, Dirac spinor fields transform according to

$$\hat{\mathcal{P}}\,\hat{\Psi}(\vec{x}, t)\,\hat{\mathcal{P}} \;\equiv\; \hat{\Psi}'(\vec{x}, t) \;=\; \pm\gamma^0\,\hat{\Psi}(-\vec{x}, t)$$

where the overall sign depends on the so-called *intrinsic parity* of a particular Dirac field. Note: $\hat{\mathcal{P}}$ here is a unitary operator in the fermionic Fock space; by nature of the parity symmetry, $\hat{\mathcal{P}}^2 = 1$.

Find the transformation rules of the bilinears under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

(3) Work carefully through section 3.5 in Peskin, completing all the steps.