Entanglement entropy, black holes and holography

Roman V. Buniy and Stephen D.H. Hsu

ITS, University of Oregon

October 11, 2005 ITS

Black hole entropy

A mystery of modern physics: $S_{\mathrm{BH}} = A$

Entropy given by area in Planck units. One bit per Planck area.

Bekenstein-Hawking: $T \sim R^{-1}$, $R \sim M$

$$S_{\mathrm{BH}} \sim \int \frac{dQ}{T} \sim \int R\, dM \sim R^2$$

The meaning of entropy

Entropy = log of number of microstates consistent with some macro condition.

$$S \sim \ln \{\text{\# of microstates } N\}$$

Typically, entropy is *extensive*:

$$\ln(c^V) = V \ln c$$

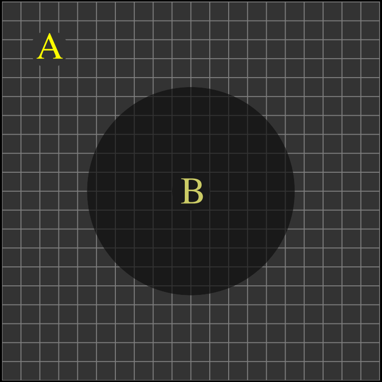
The dimensionality of the Hilbert space describing a volume V is $\dim\mathcal{H} = c^V$ ($c = 2$ for qubit). $S \sim$ number of d.o.f.

- coarse graining
- loss of information

Entropy bounds and holography

't Hooft '93, Susskind '95.

Susskind gedanken construction: let B collapse to black hole.



Entropy bound: $S_{\mathrm{B}} < A$

$S_{\mathrm{B}} \sim \ln\{\#\text{ of microstates } N\}$?

entropy not extensive?

Does gravity reduce dimensionality from d to $d-1$? AdS/CFT

Entanglement entropy

"Pure" quantum state $|\psi\rangle$: no entropy

Density matrix for "mixed" state: $\rho = \sum_n \lambda_n |n\rangle\langle n|$

von Neumann entropy: $S = -\mathrm{tr}\ \rho \ln \rho = -\sum_n \lambda_n \ln \lambda_n$

Given pure state $|\psi_{\mathrm{AB}}\rangle$, trace over region B, get density matrix

$$\rho_{n_{\mathrm{A}}\, n_{\mathrm{A}'}} = \mathrm{tr}_{\,\mathrm{B}}\, |\psi_{\mathrm{AB}}\rangle\langle\psi_{\mathrm{A}'\mathrm{B}'}| \equiv \rho_{\mathrm{A}}$$

Resulting entropy may be non-zero if $|\psi_{\mathrm{AB}}\rangle$ has correlations between A, B states.

Entanglement entropy

Simple example. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle - |--\rangle\right) ,$$

and define

$$\rho = |\psi\rangle\langle\psi| \, .$$

Then

$$\rho_1 \;=\; \mathrm{tr}_2\,\rho \;=\; {}_2\langle+|\psi\rangle\,\langle\psi|+\rangle_2 \;+\; {}_2\langle-|\psi\rangle\,\langle\psi|-\rangle_2$$

or

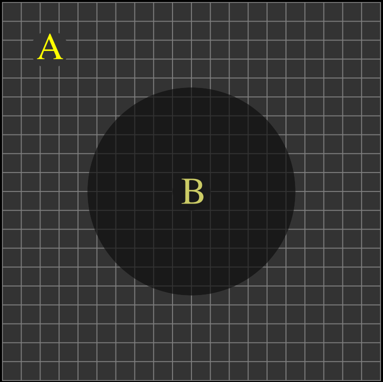
$$\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

so $S_1 = -\mathrm{tr}\,\rho_1 \ln \rho_1 = \ln 2$.

Entropy and area

Sorkin et al. (BKLS) ’87, Srednicki ’93

$|\psi_{AB}\rangle$ = QFT groundstate (e.g., free scalar field theory). Trace over region B.

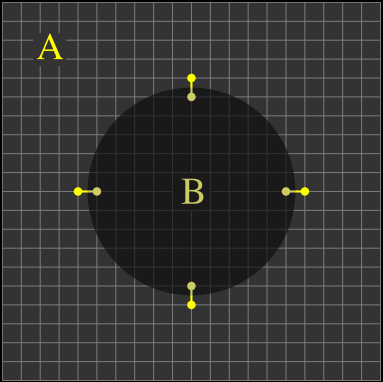


Find $S_A \sim A$! Why? How general?

Entropy and area

Sorkin et al. (BKLS) '87, Srednicki '93

$|\psi_{AB}\rangle$ = QFT groundstate (e.g., free scalar field theory). Trace over region B.



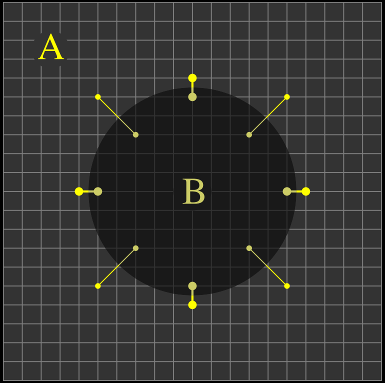
Find $S_A \sim A$! Why? How general?

A general property of quantum states with short-distance correlations.

Entropy and area

Sorkin et al. (BKLS) ’87, Srednicki ’93

$|\psi_{AB}\rangle$ = QFT groundstate (e.g., free scalar field theory). Trace over region B.



Find $S_A \sim A$! Why? How general?

A general property of quantum states with short-distance correlations.

Only $\sim c^A$ states near boundary!

Black holes?

What is the relation between entanglement entropy and black hole entropy?

$$S_{BH} = S_E + S'$$

(S', e.g., from coarse-graining location of horizon ~ Area.)

Israel '76, Einhorn et al. '05: $S_{BH} = S_E$!

At least S_E is well-defined: calculable and concrete...

Schmidt decomposition theorem

Suppose $|\psi_{\rm AB}\rangle$ is a pure state of a composite system AB. Then there exist orthonormal states $|\psi_{\rm A}^{(n)}\rangle$ for system A and $|\psi_{\rm B}^{(n)}\rangle$ for system B such that

$$|\psi_{\rm AB}\rangle = \sum_n \lambda_n^{\frac{1}{2}} \, |\psi_{\rm A}^{(n)}\rangle|\psi_{\rm B}^{(n)}\rangle\,,$$

where $\lambda_n^{\frac{1}{2}}$ are nonnegative real numbers satisfying $\sum_n \lambda_n = 1$.

- Simple consequence of the singular value decomposition theorem.

- Note dimensionalities of $\mathcal{H}_{\rm A}$ and $\mathcal{H}_{\rm B}$ might be different; range of sum determined by smaller Hilbert space.

Schmidt decomposition theorem: consequences

$$|\psi_{\mathrm{AB}}\rangle = \sum_n \lambda_n^{\frac{1}{2}} \, |\psi_{\mathrm{A}}^{(n)}\rangle |\psi_{\mathrm{B}}^{(n)}\rangle$$

$$\rho_{\mathrm{B}} = \mathrm{tr}_{\mathrm{A}} \, |\psi_{\mathrm{AB}}\rangle\langle\psi_{\mathrm{AB}}| = \sum_{n=1} \lambda_n \, |\psi_{\mathrm{B}}^{(n)}\rangle\langle\psi_{\mathrm{B}}^{(n)}|$$

$$\rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}} \, |\psi_{\mathrm{AB}}\rangle\langle\psi_{\mathrm{AB}}| = \sum_{n=1} \lambda_n \, |\psi_{\mathrm{A}}^{(n)}\rangle\langle\psi_{\mathrm{A}}^{(n)}|$$

- ρ_A and ρ_B have identical (non-zero) eigenvalues λ_n.

- Hence, $S_A = S_B$. Same entropy from tracing over small, interior region B as from tracing over large, exterior region A!

- Erroneous argument in literature: $S_A = S_B$, so entropy depends only on common AB boundary, yielding $S = f(\text{Area})$.

Entropy and volume

Search for counterexample. Can we build states $|\psi_{AB}\rangle$ with $S_E \sim V$?

"Purification" construction. Desired density matrix

$$\rho_{\mathrm{B}} = N^{-1} \sum_{n=1}^{N} |\psi_{\mathrm{B}}^{(n)}\rangle\langle\psi_{\mathrm{B}}^{(n)}|.$$

Maximal entanglement entropy $S_{\mathrm{B}} = \ln N = V \ln c$.
Complementary ρ_{A} obtained by replacing B with A in above.

We can construct a pure state; yields ρ_{A} after tracing:

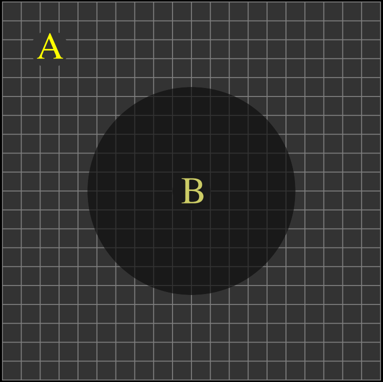
$$|\psi_{\mathrm{AB}}\rangle = N^{-\frac{1}{2}} \sum_{n=1}^{N} |\psi_{\mathrm{A}}^{(n)}\rangle\,|\psi_{\mathrm{B}}^{(n)}\rangle$$

Exterior modes can be QFT modes, qubits, etc. Correlations over distances $\sim R$.

Paradox?

Spread exterior states $|\psi_A^{(n)}\rangle$ (qubits) thinly around the universe. Let B collapse to black hole.

Usual semi-classical Hawking calculation applies, yielding $S_{\mathrm{BH}} \sim A$.



But,

$$S_{\mathrm{BH}} = S_E + S' \; ,$$

where $S' > 0$ and $S_E \sim V$.

Resolution

Gravitational collapse limits the number of states N we can use in our construction of $|\psi_{\mathrm{AB}}\rangle$. Note, subtly different from limiting the actual size of $\mathcal{H}_{\mathrm{B}}$ (holography).

In the gedanken construction, require that $|\psi_{\mathrm{AB}}\rangle$ not have already undergone collapse to a black hole larger than V. Roughly,

$$E_n < R$$

for all states n used in construction.

Resolution

't Hooft: number of states with energy less than R scales as $N_* \sim e^{A^{3/4}}$, or

$$S_* \sim \ln N_* \sim A^{3/4}.$$

Proof: Replace system under study with thermal one.

Number of states of a system with constant total energy E is given to high accuracy by the thermal result in the large volume limit (microcanonical ensemble).

Given a thermal region of radius R and temperature T, we have $S_{\text{th}} \sim T^3 R^3$ and $E \sim T^4 R^3$. Requiring $E < R$ then implies $S_{\text{th}} < R^{\frac{3}{2}} \sim A^{\frac{3}{4}}$.

In d dimensions: $S < A^{(d-1)/d}$

Resolution

Largest N we can use in our maximum entropy construction is $N \sim N_*$, which results in entropy $S_E \sim A^{3/4}$.

$$|\psi_{\mathrm{AB}}\rangle = N^{-\frac{1}{2}} \sum_{n=1}^{N} |\psi_{\mathrm{A}}^{(n)}\rangle \, |\psi_{\mathrm{B}}^{(n)}\rangle$$

We cannot violate the area scaling without producing a much larger black hole in the process (contrary to initial assumption about B geometry).

But, no truncation of Hilbert space is required. Self-consistency of gravity (no-collapse) is sufficient.

Theorem

The conclusion that collapse limits entanglement entropy is more general than our original construction.

Previously, began with states of maximal entropy and subsequently imposed no-gravitational collapse condition. Instead, maximize the entropy subject to the collapse condition.

Resulting density matrices are canonical ensembles, with Boltzmann probabilities, in contrast to the equal probabilities in our ansatz.

Nevertheless, the resulting upper bound on entropy scales only as $A^{\frac{3}{4}}$.

Theorem

Let $|\psi_{\mathrm{AB}}\rangle$ be an arbitrary pure state. Consider gravitational collapse in the region B. Whether collapse occurs depends on local properties in B, so can trace over the A degrees of freedom and consider resulting density matrix

$$\rho_{\mathrm{B}} = \sum_{n=1}^{N} \lambda_n |\psi_{\mathrm{B}}^{(n)}\rangle\langle\psi_{\mathrm{B}}^{(n)}|.$$

The bound we derive on S could either be interpreted as a bound on entanglement entropy, or simply a bound on the usual von Neumann entropy of the state which collapses to form the black hole.

No-collapse criteria (semi-classical?)

$$\mathrm{tr}(\rho_{\mathrm{B}} H_{\mathrm{B}}) = \langle H_{\mathrm{B}} \rangle < R.$$

Theorem

Maximize

$$S = -\sum_n \lambda_n \ln \lambda_n,$$

with $\{\lambda_n\}$ subject to constraints $\sum_n \lambda_n = 1$ and

$$\sum_n \lambda_n \epsilon_n = A,$$

where $\epsilon_n = E_{\mathrm{B}}^{(n)} R$. We impose equality above, since entropy will be maximized when the total energy of the system is maximal. Use method of Lagrange multipliers with

$$\tilde{S} = -\sum_n \lambda_n \ln \lambda_n + \alpha(\sum_n \lambda_n - 1) - \beta(\sum_n \lambda_n \epsilon_n - A).$$

Theorem

We obtain,

$$\lambda_n = Z^{-1} e^{-\beta \epsilon_n},$$

with

$$Z(\beta) = \sum_n e^{-\beta \epsilon_n},$$

and

$$\max S \approx A^{\frac{3}{4}}.$$

Extremal system is thermal. $T = \beta^{-1}$ is determined by average total energy condition $\langle E \rangle = R$, so result inevitably agrees with 't Hooft's calculation described earlier. One easily generalizes to d dimensions to obtain $A^{(d-1)/d}$ scaling.

Comment: $A^{3/4}$ vs A

We assumed simple boundary conditions (appropriate to a finite box) in our calculation.

Sufficient to count states which might contribute extensively (as V) to the entanglement entropy via long range correlations, but does not properly treat short range correlations at the boundary of B ($\exists$ corrections to S of order A).

Gravitational collapse condition reduces maximal contribution of “bulk” states (those not localized near the boundary) from V to $A^{\frac{3}{4}}$: smaller than the original A scaling from boundary correlations.

Conclusions

The holographic conjecture makes the rather strong assertion that states with $\langle i|H|i\rangle$ greater than R simply *do not exist* in the Hilbert space.

But why does the universe appear to have d spacetime dimensions if the Hilbert space is only that of a $d-1$ dimensional system? What happens to unitarity, locality, etc. ?

Conclusions

We suggest an alternative interpretation of black hole entropy bounds.

The gravitational collapse condition on $|\psi_{\mathrm{AB}}\rangle$ places an upper bound on the entanglement (or v.N.) entropy that can be realized from a region B without forming a black hole larger than B itself.

Highly energetic states remain in the theory, but cannot be used to increase the entropy beyond the area of B in Planck units.

Entropy bounds reflect the limitations that gravity imposes on the construction of pure states or density matrices, but do not require a truncation of the Hilbert space itself.