

# Entanglement entropy, black holes and holography

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# Black hole entropy

A mystery of modern physics:  $S_{\text{BH}} = A/4$

Entropy given by area in Planck units. One bit per Planck area.

Bekenstein-Hawking:  $T \sim R^{-1}$  ,  $R \sim M$

$$S_{\text{BH}} \sim \int \frac{dQ}{T} \sim \int R dM \sim R^2$$

# The meaning of entropy

Entropy = log of number of microstates consistent with some macro condition.

$$S \sim \ln \{\# \text{ of microstates } N\}$$

Typically, entropy is *extensive*:

$$\ln(c^V) = V \ln c$$

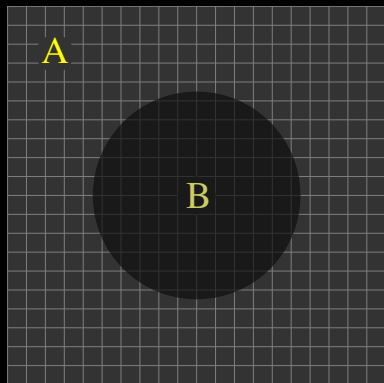
The dimensionality of the Hilbert space describing a volume  $V$  is  $\dim \mathcal{H} = c^V$  ( $c = 2$  for qubit).  $S \sim$  number of d.o.f.

- coarse graining
- loss of information

# Entropy bounds and holography

't Hooft '93, Susskind '95.

Susskind gedanken construction: let  $B$  collapse to black hole.



Entropy bound:  $S_B < \text{Area}$

$S_B \sim \ln \{\text{\# of microstates } N\}$  ?

entropy not extensive?

Does gravity reduce dimensionality from  $d$  to  $d - 1$ ? AdS/CFT

# Entanglement entropy

“Pure” quantum state  $|\psi\rangle$ : no entropy

Density matrix for “mixed” state:  $\rho = \sum_n \lambda_n |n\rangle\langle n|$

von Neumann entropy:  $S = -\text{tr } \rho \ln \rho = -\sum_n \lambda_n \ln \lambda_n$

Given pure state  $|\psi_{AB}\rangle$ , trace over region B, get density matrix

$$\rho_{n_A n_A'} = \text{tr}_B |\psi_{AB}\rangle\langle\psi_{A'B'}| \equiv \rho_A$$

Resulting entropy may be non-zero if  $|\psi_{AB}\rangle$  has correlations between A, B states.

# Entanglement entropy

Simple example. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|++\rangle - |--\rangle),$$

and define

$$\rho = |\psi\rangle\langle\psi|.$$

Then

$$\rho_1 = \text{tr}_2 \rho = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)$$

or

$$\rho_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

so  $S_1 = -\text{tr} \rho_1 \ln \rho_1 = \ln 2$ .

## Two qubits

$$\psi_{AB} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B + b \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B + c \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B + d \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B$$

$$\rho_{AB} = aa^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_B + ab^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_B + \dots$$

$$\rho_A = \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ ca^* + db^* & cc^* + dd^* \end{pmatrix}_A, \quad \rho_B = \begin{pmatrix} aa^* + cc^* & ab^* + cd^* \\ ba^* + dc^* & bb^* + dd^* \end{pmatrix}_B$$

$$\lambda_{1,2} = \frac{1}{2} \left[ 1 \pm \left( 1 - 4|ad - bc|^2 \right)^{1/2} \right] \Rightarrow S_A = S_B = -\sum_n \lambda_n \ln \lambda_n$$

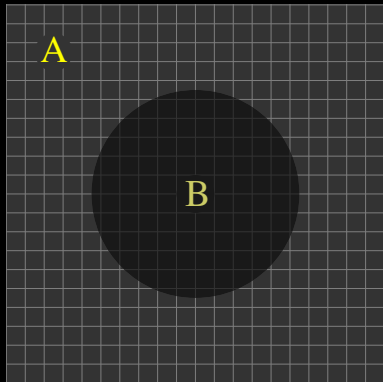
$ad - bc = 0$  : pure state,  $S = 0$

$ad - bc \neq 0$  : mixed state,  $S > 0$

# Entropy and area

Sorkin et al. (BKLS) '87, Srednicki '93

$|\psi_{AB}\rangle =$  QFT groundstate (e.g., free scalar field theory). Trace over region B.

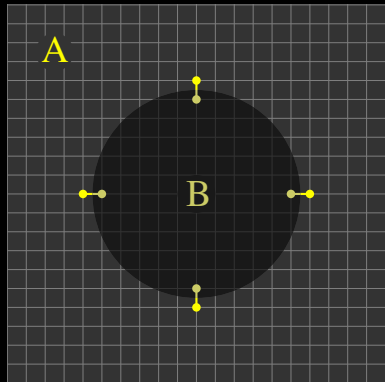


Find  $S_A \sim \text{Area}$  ! Why?  
How general?

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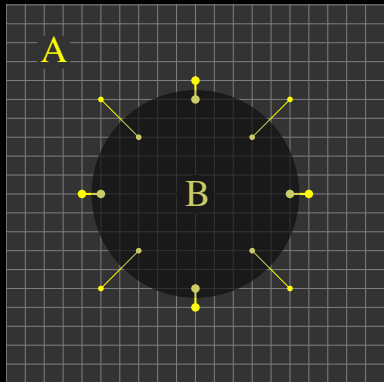
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A general property of  
quantum states with  
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Find  $S_A \sim \text{Area}$  ! Why?  
How general?

A general property of quantum states with short-distance correlations.

Only  $\sim c^{\text{Area}}$  states near boundary!

# Black holes?

What is the relation between entanglement entropy and black hole entropy?

$$S_{BH} = S_E + S'$$

( $S'$ , e.g., from coarse-graining location of horizon  $\sim$  Area.)

Israel '76, Einhorn et al. '05:  $S_{BH} = S_E$  !

At least  $S_E$  is well-defined: calculable and concrete...

# Schmidt decomposition theorem

Suppose  $|\psi_{AB}\rangle$  is a pure state of a composite system AB. Then there exist orthonormal states  $|\psi_A^{(n)}\rangle$  for system A and  $|\psi_B^{(n)}\rangle$  for system B such that

$$|\psi_{AB}\rangle = \sum_n \lambda_n^{\frac{1}{2}} |\psi_A^{(n)}\rangle |\psi_B^{(n)}\rangle,$$

where  $\lambda_n^{\frac{1}{2}}$  are nonnegative real numbers satisfying  $\sum_n \lambda_n = 1$ .

- Simple consequence of the singular value decomposition theorem.
- Note dimensionalities of  $\mathcal{H}_A$  and  $\mathcal{H}_B$  might be different; range of sum determined by smaller Hilbert space.

# Schmidt decomposition theorem: consequences

$$|\psi_{AB}\rangle = \sum_n \lambda_n^{\frac{1}{2}} |\psi_A^{(n)}\rangle |\psi_B^{(n)}\rangle$$

$$\rho_B = \text{tr}_A |\psi_{AB}\rangle \langle \psi_{AB}| = \sum_{n=1} \lambda_n |\psi_B^{(n)}\rangle \langle \psi_B^{(n)}|$$

$$\rho_A = \text{tr}_B |\psi_{AB}\rangle \langle \psi_{AB}| = \sum_{n=1} \lambda_n |\psi_A^{(n)}\rangle \langle \psi_A^{(n)}|$$

- $\rho_A$  and  $\rho_B$  have **identical** (non-zero) eigenvalues  $\lambda_n$ .
- Hence,  $S_A = S_B$ . **Same entropy** from tracing over small, interior region B as from tracing over large, exterior region A!
- **Erroneous** argument in literature:  $S_A = S_B$ , so entropy depends only on common AB boundary, yielding  $S = f(\text{Area})$ .

# Entropy and volume

Search for counterexample. Can we build states  $|\psi_{AB}\rangle$  with  $S_E \sim V$ ?

“Purification” construction. Desired density matrix

$$\rho_B = N^{-1} \sum_{n=1}^N |\psi_B^{(n)}\rangle\langle\psi_B^{(n)}|.$$

Maximal entanglement entropy  $S_B = \ln N = V \ln c$ .

Complementary  $\rho_A$  obtained by replacing B with A in above.

We can construct a pure state; yields  $\rho_A$  after tracing:

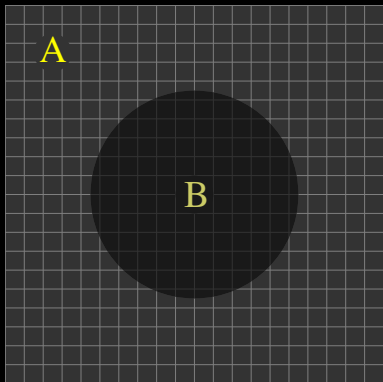
$$|\psi_{AB}\rangle = N^{-\frac{1}{2}} \sum_{n=1}^N |\psi_A^{(n)}\rangle |\psi_B^{(n)}\rangle$$

Exterior modes can be QFT modes, qubits, etc. Correlations over distances  $\sim R$ .

# Paradox?

Spread exterior states  $|\psi_A^{(n)}\rangle$  (qubits) thinly around the universe.  
Let  $B$  collapse to black hole.

Usual semi-classical Hawking calculation applies, yielding  
 $S_{\text{BH}} \sim A$ .



But,

$$S_{\text{BH}} = S_E + S' ,$$

where  $S' > 0$  and  $S_E \sim V$ .

# Resolution

Gravitational collapse limits the number of states  $N$  we can use in our construction of  $|\psi_{AB}\rangle$ . Note, subtly different from limiting the actual size of  $\mathcal{H}_B$  (holography).

In the gedanken construction, require that  $|\psi_{AB}\rangle$  not have already undergone collapse to a black hole larger than  $V$ . Roughly,

$$E_n < R$$

for all states  $n$  used in construction.

# Resolution

't Hooft: number of states with energy less than  $R$  scales as  $N_* \sim e^{A^{3/4}}$ , or

$$S_* \sim \ln N_* \sim A^{3/4}.$$

**Proof:** Replace system under study with thermal one.

Number of states of a system with constant total energy  $E$  is given to high accuracy by the thermal result in the large volume limit (microcanonical ensemble).

Given a thermal region of radius  $R$  and temperature  $T$ , we have  $S_{\text{th}} \sim T^3 R^3$  and  $E \sim T^4 R^3$ . Requiring  $E < R$  then implies  $S_{\text{th}} < R^{3/2} \sim A^{3/4}$ .

In  $d$  dimensions:  $S < A^{(d-1)/d}$

# Resolution

Largest  $N$  we can use in our maximum entropy construction is  $N \sim N_*$ , which results in entropy  $S_E \sim A^{3/4}$ .

$$|\psi_{AB}\rangle = N^{-\frac{1}{2}} \sum_{n=1}^N |\psi_A^{(n)}\rangle |\psi_B^{(n)}\rangle$$

We cannot violate the area scaling without producing a much larger black hole in the process (contrary to initial assumption about B geometry).

But, no truncation of Hilbert space is required. Self-consistency imposed by gravity (no-collapse) is sufficient.

# Theorem

The conclusion that collapse limits entanglement entropy is more general than our original construction.

Previously, began with states of maximal entropy and subsequently imposed no-gravitational collapse condition. **Instead, maximize the entropy subject to the collapse condition.**

**Resulting density matrices are canonical ensembles, with Boltzmann probabilities, in contrast to the equal probabilities in our ansatz.**

**Nevertheless, the resulting upper bound on entropy scales only as  $A^{\frac{3}{4}}$ .**

# Theorem

Let  $|\psi_{AB}\rangle$  be an arbitrary pure state. Consider gravitational collapse in the region B. Whether collapse occurs depends on local properties in B, so can trace over the A degrees of freedom and consider resulting density matrix

$$\rho_B = \sum_{n=1}^N \lambda_n |\psi_B^{(n)}\rangle \langle \psi_B^{(n)}|.$$

The bound we derive on  $S$  could either be interpreted as a bound on entanglement entropy, or simply a bound on the usual von Neumann entropy of the state which collapses to form the black hole.

No-collapse criteria (semi-classical?)

$$\text{tr}(\rho_B H_B) = \langle H_B \rangle < R.$$

# Theorem

Maximize

$$S = - \sum_n \lambda_n \ln \lambda_n,$$

with  $\{\lambda_n\}$  subject to constraints  $\sum_n \lambda_n = 1$  and

$$\sum_n \lambda_n \epsilon_n = A,$$

where  $\epsilon_n = E_B^{(n)} R$ . We impose equality above, since entropy will be maximized when the total energy of the system is maximal. Use method of Lagrange multipliers with

$$\tilde{S} = - \sum_n \lambda_n \ln \lambda_n + \alpha \left( \sum_n \lambda_n - 1 \right) - \beta \left( \sum_n \lambda_n \epsilon_n - A \right).$$

# Theorem

We obtain,

$$\lambda_n = Z^{-1} e^{-\beta \epsilon_n},$$

with

$$Z(\beta) = \sum_n e^{-\beta \epsilon_n},$$

and

$$\max S \approx A^{\frac{3}{4}}.$$

Extremal system is thermal.  $T = \beta^{-1}$  is determined by average total energy condition  $\langle E \rangle = R$ , so **result inevitably agrees with 't Hooft's calculation described earlier**. One easily generalizes to  $d$  dimensions to obtain  $A^{(d-1)/d}$  scaling.

## Comment: $A^{3/4}$ vs $A$

We assumed **simple boundary conditions** (appropriate to a finite box) in our calculation.

**Sufficient to count states which might contribute extensively** (as  $V$ ) to the entanglement entropy via long range correlations, but does not properly treat short range correlations at the boundary of  $B$  ( $\exists$  corrections to  $S$  of order  $A$ ).

Gravitational collapse condition reduces maximal contribution of “bulk” states (those not localized near the boundary) from  $V$  to  $A^{3/4}$ : smaller than the original  $A$  scaling from boundary correlations.

# Information processing

S.H, Phys.Lett.B641:99, hep-th/0607082.

Margolus-Levitin:  $\Delta t > \frac{\pi}{2} E^{-1}$

ML + hoop conjecture: geometric bound on information processing rate

$$\mathcal{R} < E < L \sim V^{1/3}$$

Paradox for Nature: evolution of physical systems requires extensive processing rate?

Saved by holography!

# Conclusions

The holographic conjecture makes the rather strong assertion that states with  $\langle i|H|i\rangle$  greater than  $R$  simply *do not exist* in the Hilbert space.

But why does the universe appear to have  $d$  spacetime dimensions if the Hilbert space is only that of a  $d - 1$  dimensional system? What happens to unitarity, locality, etc. ?

# Conclusions

We suggest an alternative interpretation of black hole entropy bounds.

The gravitational collapse condition on  $|\psi_{AB}\rangle$  places an upper bound on the entanglement (or v.N.) entropy that can be realized from a region B without forming a black hole larger than B itself.

Highly energetic states remain in the theory, but cannot be used to increase the entropy beyond the area of B in Planck units.

Entropy bounds reflect the limitations that gravity imposes on the construction of pure states or density matrices, but do not require a truncation of the Hilbert space itself.