Positivity of Fermi surface effective theory

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D.K. Hong and S. Hsu, Phys.Rev.D66:071501,2002 (hep-ph/0202236), Phys.Rev.D68:034011,2003 (hep-ph/0304156) and LATTICE 2003 (hep-lat/0309103).

Sign Problem in QCD at finite density. Precludes lattice simulation and rigorous theorems. (Also an issue in condensed matter: many body simulations.)

How fundamental? fermions = sign problem? But, zero-density QCD has no sign problem.

Finite density $\longleftrightarrow$ Fermi surface. But we will show there is No Sign Problem for excitations near the Fermi surface. Use HDET = High Density Effective Theory.

Intuition: curvature of FS not evident to low energy modes. Fermi sea looks like Dirac sea.

This observation leads to possible methods for lattice simulation, and some rigorous theorems for QCD at asymptotic density.

Long term prospects? Recall lattice chiral fermions.

Euclidean Path Integrals and Importance Sampling

Partition function

$$Z = \int \prod_{\{x\}} dA_x \; d\bar{\psi}_x \; d\psi_x \;\; e^{-\beta S[A,\bar{\psi},\psi]}$$

Multi-dimensional integral ($> 10^8$!) cannot be evaluated directly.

Use techniques like Monte Carlo with importance sampling.

For importance sampling, want integrand which can be interpreted as probability distribution (real, positive).

Worst case:

$$Z = \sum (\; + \;\; - \;\; + \;\; - \;\; + \;\; - \;\; + \;\; \cdots)$$

Luckily, lattice QCD integrand positive and real at zero baryon density. But not true at finite chemical potential.

Simple example: (1+1) Dimensions

• Euclidean (1+1) action of non-relativistic fermions interacting with a gauge field A

$$S \;=\; \int d\tau dx\; \psi^*_\sigma \left[(-\partial_\tau + i\phi + \epsilon_F) \;-\; \epsilon(-i\partial_x + A)\right] \psi_\sigma \qquad (1)$$

where $\epsilon(p)$ is the energy as a function of momentum (e.g. $\epsilon(p) \approx \frac{p^2}{2m} + \cdots$).

• Dispersion relation with chemical potential: $E(p) = \epsilon(p) - \epsilon_F$. Low energy modes have momentum near $\pm p_F$ ($\epsilon(\pm p_F) = \epsilon_F$).

• (1+1) Fermi surface in (1+1): two points $p = \pm p_F$. Near these points

$$E(p \pm p_F) \;\approx\; \pm\; v_F p \quad , \qquad (2)$$

where $v_F = \partial E/\partial p|_{p_F}$ is the Fermi velocity.

• Action (1) not obviously positive. Operator in brackets $[\;\cdots\;]$ has complex eigenvalues.

Assume gauge field has small amplitude and is slowly varying relative to scale p_F. Extract the slowly varying component of the fermion $\longrightarrow$ low energy effective theory involving quasiparticles and gauge fields with positive, semi-definite determinant.

• Extract quasiparticle modes:

$$\psi(x,\tau) \;=\; \psi_L e^{+ip_F x} \;+\; \psi_R e^{-ip_F x} \quad , \tag{3}$$

where $\psi_{L,R}$ are slowly varying. Use

$$e^{\pm ip_F x}\, E(-i\partial_x + A)\, e^{\mp ip_F x}\, \psi(x) \approx \pm\, v_F(-i\partial_x + A)\psi(x) \quad , \tag{4}$$

to obtain

$$\begin{aligned} S_{\text{eff}} \;&=\; \int d\tau\, dx\, [\psi_L^*(-\partial_\tau + i\phi + i\partial_x - A)\psi_L \\ &+\; \psi_R^*(-\partial_\tau + i\phi - i\partial_x + A)\psi_R]. \end{aligned} \tag{5}$$

• Introduce the Euclidean (1+1) gamma matrices $\gamma_{0,1,2}$, which are Hermitian and can be taken as $\gamma_i = \sigma_i$ where $\vec{\sigma}$ are the Pauli matrices. Using $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_2)\psi$ we obtain

$$S_{\text{eff}} = \int d\tau dx\; \bar{\psi}\gamma^\mu(\partial_\mu + iA_\mu)\psi \;\equiv\; \int d\tau dx\; \bar{\psi}\not{D}\psi \quad . \tag{6}$$

• Since the gamma matrices are Hermitian, and the operator $(\partial_\mu + iA_\mu)$ is anti-Hermitian, the operator $\not{D}$ in (6) has purely imaginary eigenvalues. Since γ_2 anticommutes with $\not{D}$, the eigenvalues come in conjugate pairs: given $\not{D}\,\phi = \lambda\phi$, we have

$$\not{D}\,(\gamma_2\phi) = -\gamma_2\not{D}\,\phi = -\gamma_2\lambda\phi = -\lambda(\gamma_2\phi_n) \quad .$$

Hence the determinant $\det\not{D} \;=\; \Pi\,\lambda^*\lambda$ is real and positive semi-definite.

• By considering only the low-energy modes near the Fermi points, we obtain an effective theory with desirable positivity properties.

• Note: interactions (background gauge field A) must not couple strongly the low-energy modes to fast modes which are far from the Fermi points. A reasonable approximation in many situations, e.g., if interactions among quasiparticles of primary interest.

RECIPE:

Near Fermi surface, modes have low energy and are slowly varying. Coupling these modes to slowly varying background field A leads to a positive effective theory.

Slowly varying = relative to Fermi momentum p_F.

QCD: strong coupling dynamics at scales $\sim \Lambda_{\rm QCD}$. By taking

$$\mu \ \sim \ p_F \ >> \ \Lambda_{\rm QCD}$$

we ensure that quark quasiparticles couple only to slowly varying, small amplitude background fields A.

Note: dense quarks are overall color neutral.

QCD at finite chemical potential:

$$\mathcal{L}_{\rm QCD} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \mu \bar{\psi} \gamma_0 \psi, \quad (7)$$

where the covariant derivative $D_\mu = \partial_\mu + i A_\mu$ and we neglect the mass of quarks for simplicity.

- Effective Fermi Surface theory (HDET)

Decompose the momentum of a quark into a Fermi momentum and a residual momentum as

$$p_\mu = \mu v_\mu + l_\mu, \quad (8)$$

Region near the Fermi surface: $l_0, |\vec{l}| << \mu$.

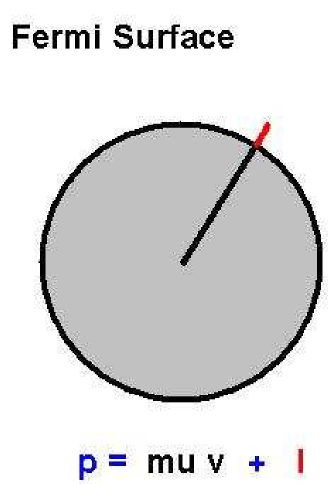


Figure 1: Fermi surface and momentum decomposition.

• In Euclidean space, the Dirac operator $i\not{D}+\mu\gamma_0$ is neither Hermitian nor anti-Hermitian, so it has complex eigenvalues $\longrightarrow$ complex fermion determinant $\longrightarrow$ No importance sampling.

• Recall why the measure of dense QCD is complex in Euclidean space. We use the following analytic continuation of the Dirac lagrangian to Euclidean space:

$$x_0 \to i x_E^4, \quad x_i \to x_E^i \; ; \; \gamma_0 \to \gamma_E^4, \quad \gamma_i \to i\gamma_E^i \quad . \tag{9}$$

Euclidean gamma matrices satisfy

$$\gamma_E^{\mu\,\dagger} = \gamma_E^\mu \quad , \quad \{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu}. \tag{10}$$

• The grand canonical partition function for QCD is

$$Z(\mu) = \int \mathrm{d}A_\mu \det\left(M\right) e^{-S(A_\mu)}, \tag{11}$$

where $S(A_\mu)$ is the positive semi-definite gauge action, and the Dirac operator

$$M \;=\; \gamma_E^\mu D_E^\mu \;+\; \mu\gamma_E^4, \tag{12}$$

where $D_E = \partial_E + iA_E$ is the analytic continuation of the covariant derivative.

• The Hermitian conjugate of the Dirac operator is

$$M^\dagger = -\gamma_E^\mu D_E^\mu \;+\; \mu\gamma_E^4 \quad . \tag{13}$$

• The first term in (12) is anti-Hermitian, while the second is Hermitian, hence the generally complex eigenvalues. When $\mu = 0$, the eigenvalues are purely imaginary, but come in conjugate pairs (λ, λ^*), so the resulting determinant is real and positive.

• To demonstrate this, note that γ_5 anti-commutes with M, so if $M\phi = \lambda\phi$, then $M\gamma_5\phi = -\gamma_5 M\phi = -\lambda\gamma_5\phi$.

HDET: High Density Effective Field Theory

Decompose the quark fields as (like Heavy Quark EFT!):

$$\psi(x) = \sum_{\vec{v}_F} \left[e^{i\mu\vec{v}_F\cdot\vec{x}}\psi_+(\vec{v}_F, x) + e^{i\mu\vec{v}_F\cdot\vec{x}}\psi_-(\vec{v}_F, x) \right], \tag{14}$$

where $\vec{\alpha}\cdot\vec{v}_F\psi_\pm(\vec{v}_F, x) = \pm\psi_\pm(\vec{v}_F, x)$ is a projector on (anti)particle states.

• We want to isolate *particle* modes near the Fermi surface.

The quark Lagrangian in (7) then becomes

$$\begin{aligned}\bar{\psi}\left(i\not{D} + \mu\gamma^0\right)\psi &= \sum_{\vec{v}_F}[\bar{\psi}_+(\vec{v}_F, x)i\gamma_\parallel^\mu D_\mu\psi_+(\vec{v}_F, x) \\ &+ \bar{\psi}_-(\vec{v}_F, x)\gamma^0\left(2\mu + iD_\parallel\right)\psi_-(\vec{v}_F, x)] \\ &+ \sum_{\vec{v}_F}[\bar{\psi}_-(\vec{v}_F, x)i\not{D}_\perp\psi_+(\vec{v}_F, x) + \text{h.c.}]\end{aligned} \tag{15}$$

where $\gamma_\parallel^\mu \equiv (\gamma^0, \vec{v}_F\vec{v}_F\cdot\vec{\gamma})$, $\gamma_\perp^\mu = \gamma^\mu - \gamma_\parallel^\mu$, $D_\parallel = \bar{V}^\mu D_\mu$ with $\bar{V}^\mu = (1, -\vec{v}_F)$, and $\not{D}_\perp = \gamma_\perp^\mu D_\mu$.

• Eliminating massive ψ_- modes, we obtain the tree-level Lagrangian for ψ_+

$$\mathcal{L}_{\rm eff}^0 = \bar{\psi}_+ i\gamma_\parallel^\mu D_\mu\psi_+ \; - \; \frac{1}{2\mu}\bar{\psi}_+\gamma^0(\not{D}_\perp)^2\psi_+ \; + \; \cdots, \tag{16}$$

where the ellipsis denotes terms with higher derivatives.

• Previous formulation involves sum over "patches" on the Fermi sphere: $\Sigma_{v_F} \; \cdots$. This introduces a redundancy of description, analogous to reparametrization invariance in HQET.

• Need a formulation suitable for functional integral. Restriction of modes (elimination of large Fermi momentum and particle projection) accomplished by operator:

$$\psi(x) = e^{iX} \psi_+(x)$$

• The leading (Euclidean) Dirac operator is of the form

$$\mathcal{L}_+ = \bar{\psi}_+ e^{-iX} \left(i\partial\!\!\!/ - A\!\!\!/ + \mu\gamma_0 \right) e^{+iX} \psi_+ \quad , \tag{17}$$

where X is a Hermitian operator which removes the "large" component of momentum $\mu \vec{v}_F$:

$$X \;\equiv\; \mu \; x \cdot v \; \alpha \cdot v \;=\; \mu \frac{\alpha^i x^j}{\nabla^2} \frac{\partial^2}{\partial x^i \partial x^j} . \tag{18}$$

We can rewrite (17) as:

$$\mathcal{L}_+ = \bar{\psi}_+ \gamma_{\|}^{\mu} \left(\partial^\mu + i A_+^\mu \right) \psi_+ \quad , \tag{19}$$

where

$$A_+^\mu = e^{-iX} \; A^\mu \; e^{+iX} \quad . \tag{20}$$

• The operator in (19) is anti-Hermitian and leads to a positive, semidefinite determinant since it anti-commutes with γ_5.

Application: lattice simulation

Back to the original (not HDET) QCD partition function:

$$Z(\mu) = \int \mathrm{d}A_\mu \ \det(M) \ e^{-S(A_\mu)} \ .$$

M is the Dirac operator at finite density, $\mathbf{A}$ the usual gauge field.

It is easy to show that at $\mathbf{A} = 0$ (zero background gauge field), the Dirac determinant is real even at finite density.

Now, consider small amplitude, slowly varying background gauge fields $\mathbf{A}$ whose magnitude and derivatives $\partial\mathbf{A}$ are small relative to μ. (e.g. $\mu >> \Lambda_{\rm QCD} \sim \mathbf{A}$, $\partial\mathbf{A}$, or $\mathbf{F}_{\mu\nu}, \mathbf{D}_\mu$.)

Expand about the FS. Integrate out heavy modes (antiquarks, quarks far from the FS). These modes contribute to $\det(M)$, but their contribution is suppressed by $1/\mu$. Find

$$\det(M) \ = \ [\text{real, positive}] \left(1 \ + \ \mathcal{O}\left(\frac{\mathbf{F}}{\mu^2}\right)\right)$$

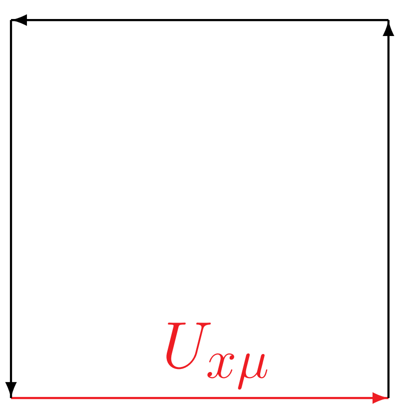
How to enforce small amplitude, slowly varying gauge field $\mathbf{A}$?

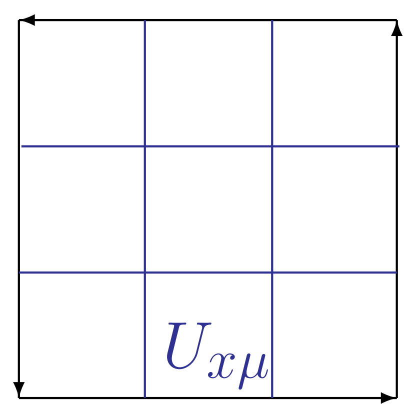
Use two lattices with different spacings $a_{\rm det}$, $a_{\rm gauge}$. Compute determinant on lattice with spacing $a_{\rm det} \sim \mu^{-1} << a_{\rm gauge}$.

Determinant is a function of plaquettes $\{\mathbf{U}_{\mathbf{x}\mu}\}$ which are obtained by interpolation from the plaquettes on the coarser a_{gauge} lattice.

Interpolation: link variables $\mathbf{U}_{\mathbf{x}\mu} \in SU(3)$. Connect any two points g_1, g_2 on the group manifold:

$$g(t) = g_1 + t(g_2 - g_1)\ ,\ 0 \le t \le 1$$





• Use leading real, positive part of determinant for importance sampling.

• Nontrivial check on analytic results at asymptotic density. Extrapolate to intermediate density?

• Temperature effects: $\mathcal{O}\left(\frac{T}{\mu}\right)$. Typical quasiparticle not exactly at FS. At finite T, there is a complex phase dependence of determinant on background constant A_0 potential.

Application: Vafa-Witten inequalities at finite density

• Can derive inequalities which govern the limit of asymptotic density. These show that CFL is the true groundstate at high density

• Vector current correlators fall off exponentially, if all quarks are gapped.

$$\left\langle J_\mu^a(x) J_\nu^b(y)\right\rangle^A = -\mathrm{Tr}\, \gamma_\mu T^a S^A(x,y;\Delta)\gamma_\nu T^b S^A(y,x;\Delta),$$

with $J_\mu^a(x) = \bar\psi_+(x)\gamma_\mu T^a \psi_+(x)$. The propagator is

$$\langle x|\,\frac{1}{M}\,|y\rangle = \int_0^\infty \mathrm{d}\tau\, \langle x|\, e^{-i\tau(-iM)}\, |y\rangle$$

where with $D = \partial + iA$

$$M = \gamma_0 \begin{pmatrix} D\cdot V & \Delta \\ \Delta & D\cdot \bar V \end{pmatrix},$$

• Eigenvalues of M bounded from below by Δ, yielding the inequality:

$$\left| \;\langle x|\,\frac{1}{M}\,|y\rangle\; \right| \;\le\; \int_R^\infty \mathrm{d}\tau\; e^{-\Delta\tau}\, \sqrt{\langle x|x\rangle}\sqrt{\langle y|y\rangle} \;=\; \frac{e^{-\Delta R}}{\Delta}\sqrt{\langle x|x\rangle}\sqrt{\langle y|y\rangle}.$$

No Goldstone mode in the vector channel. For three massless flavors $SU(3)_V$ is unbroken $\to$ CFL phase.

N.B. When quark masses are non-zero one has to study the $\mu \to \infty$ and $m_q \to 0$ limit carefully to see whether the inequalities apply. If m_s goes to zero too slowly, can get Kaon condensation, which breaks isospin.

Conclusion and Future Prospects

• (Euclidean) Sign problem may not be as generic as we think.

• Lattice simulation of high density matter using importance sampling is possible. We can use the ordinary quark determinant if we are willing to interpolate the gauge plaquettes to a finer lattice.

• One can show rigorously that CFL is the true vacuum state of 3 flavor massless QCD at asymptotic density.

• There may be applications to condensed matter systems. (High T_c, Hubbard-like models?)