Entanglement entropy, black holes and holography

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Outline

Entropy

- $\mathcal{N}$ microscopic degrees of freedom consistent with a macroscopic constraint
- entropy: $S = \ln \mathcal{N}$
- usually entropy is extensive: $S \sim V$
- Hilbert space of dimension $\dim \mathcal{H} \sim e^V \sim \mathcal{N}$

- a closed system in statistical equilibrium
- $\lambda_n^{(a)}$: probability for subsystem a to be in a state n
- entropy of subsystem a: $S^{(a)} = -\sum_n \lambda_n^{(a)} \log \lambda_n^{(a)}$
- total entropy: $S = \sum_a S^{(a)}$

Black hole entropy

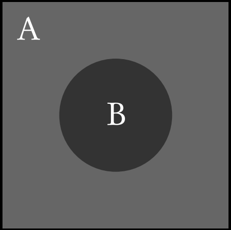
A mystery of modern physics: $S_{\mathrm{BH}} \sim A$

Bekenstein-Hawking: $T \sim R^{-1}$, $R \sim M$

$$S_{\mathrm{BH}} \sim \int \frac{dQ}{T} \sim \int R\, dM \sim R^2$$

Susskind's argument

Gedanken construction: let B + additional shell collapse to black hole.



Entropy bound: $S_B <$ Area
(not extensive)

$S_B = \ln \mathcal{N}_B$?

Does gravity reduce dimensionality from d to $d - 1$? AdS/CFT

’t Hooft’s argument

How many QFT states in a region of size R and energy E?

- the most probable state: gas at some temperature T
- energy: $E \sim R^3 T^4$
- entropy: $S_{\text{th}} \sim R^3 T^3$
- no gravitational collapse: $E \lesssim R$
- temperature: $T \lesssim R^{-1/2}$
- entropy: $S_{\text{th}} \lesssim R^{3/2} \sim A^{3/4}$
- a single black hole ($S_{\text{th}} \sim A$) is the limit
- degrees of freedom: $\dim\mathcal{H} = \mathcal{N} \lesssim e^{A}$

Radical proposal: to describe physics in V it is enough to have one degree of freedom (one bit of information) per Planck area of ∂V.

Entropy and area

Sorkin et al. '87, Srednicki '93

Model: free field theory

Two regions: A and B

Wavefunction: ψ_{AB} (QFT ground state)

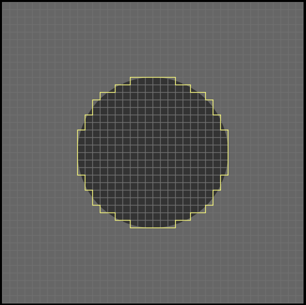
Trace over region B

Find $S_{\mathrm{E}}(\mathrm{A}) \sim$ Area (entanglement entropy)

A general property of any theory with short-distance correlations

Only $\mathcal{N} \sim e^{A}$ states near the boundary

Black holes?



What is the relation between entanglement entropy and black hole entropy?

$$S_{\mathrm{BH}} = S_{\mathrm{E}} + S'$$

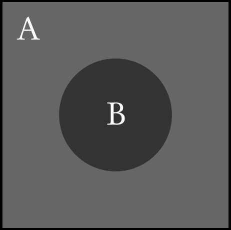
($S' > 0$, e.g., from coarse-graining location of the horizon $\sim$ Area.)

Israel '76, Einhorn et al. '05: $S_{\mathrm{BH}} = S_{\mathrm{E}}$

(At least S_{E} is well-defined (calculable and concrete) and it should be a component of S_{BH}.)

Pure and mixed states

Eigenfunction expansion: $\psi_{\rm AB} = \sum_{mn} C_{mn} \psi_{\rm A}^{(m)} \psi_{\rm B}^{(n)} \quad (C : \mathcal{M} \times \mathcal{N})$



Pure states: $C_{mn} = C_m C_n$
Mixed states: $C_{mn} \neq C_m C_n$

Density matrix: $\rho_{\rm AB} = \psi_{\rm AB} \psi_{\rm AB}^{\dagger}$

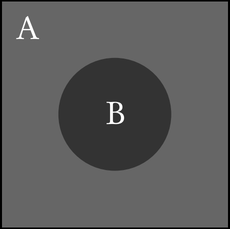
$\rho_{\rm AB}$ is pure iff $\rho_{\rm AB}^2 = \rho_{\rm AB}$

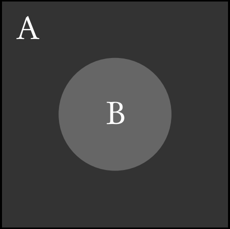
Averages: $\langle O_{\rm AB} \rangle_{\rm AB} = {\rm tr}_{\rm AB}(\rho_{\rm AB} O_{\rm AB})$

Entropy: $S_{\rm AB} = -{\rm tr}_{\rm AB} \rho_{\rm AB} \ln \rho_{\rm AB} = \begin{cases} 0 & \text{for pure states} \\ > 0 & \text{for mixed states} \end{cases}$

Reduced density matrices

pure ρ_{AB}





$$\begin{aligned} \rho_A &= \mathrm{tr}_B\, \rho_{AB} \\ S_A &= -\mathrm{tr}_A\, \rho_A \ln \rho_A \end{aligned}$$

$$\begin{aligned} \rho_B &= \mathrm{tr}_A\, \rho_{AB} \\ S_B &= -\mathrm{tr}_B\, \rho_B \ln \rho_B \end{aligned}$$

$S_A = S_B$

(not immediately obvious)

Two qubits

$$\psi_{AB} = a\begin{pmatrix}1\\0\end{pmatrix}_A\begin{pmatrix}1\\0\end{pmatrix}_B + b\begin{pmatrix}1\\0\end{pmatrix}_A\begin{pmatrix}0\\1\end{pmatrix}_B + c\begin{pmatrix}0\\1\end{pmatrix}_A\begin{pmatrix}1\\0\end{pmatrix}_B + d\begin{pmatrix}0\\1\end{pmatrix}_A\begin{pmatrix}0\\1\end{pmatrix}_B$$

$$\rho_{AB} = aa^*\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}_A\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}_B + ab^*\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}_A\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}_B + \cdots$$

$$\rho_A = \begin{pmatrix}aa^* + bb^* & ac^* + bd^*\\ ca^* + db^* & cc^* + dd^*\end{pmatrix}_A, \quad \rho_B = \begin{pmatrix}aa^* + cc^* & ab^* + cd^*\\ ba^* + dc^* & bb^* + dd^*\end{pmatrix}_B$$

$$\lambda_{1,2} = \frac{1}{2}\left[1 \pm \left(1 - 4|ad - bc|^2\right)^{1/2}\right] \quad \Rightarrow \quad S_A = S_B = -\sum_n \lambda_n \ln \lambda_n$$

$ad - bc = 0 :$ pure state, $S = 0$

$ad - bc \neq 0 :$ mixed state, $S > 0$

Proof of $S_{\mathrm{A}} = S_{\mathrm{B}}$

Eigenfunction expansion: $\psi_{\mathrm{AB}} = \sum_{mn} C_{mn} \psi_{\mathrm{A}}^{(m)} \psi_{\mathrm{B}}^{(n)} \quad (C : \mathcal{M} \times \mathcal{N})$

Reduced density matrices:

$$\begin{aligned} \rho_{\mathrm{A}} &= \mathrm{tr}_{\mathrm{B}} \psi_{\mathrm{AB}} \psi_{\mathrm{AB}}^{\dagger} = \sum_{mm'} (CC^{\dagger})_{mm'} \psi_{\mathrm{A}}^{(m)} \psi_{\mathrm{A}}^{(m')\dagger} \\ \rho_{\mathrm{B}} &= \mathrm{tr}_{\mathrm{A}} \psi_{\mathrm{AB}} \psi_{\mathrm{AB}}^{\dagger} = \sum_{nn'} (C^{\dagger}C)_{n'n} \psi_{\mathrm{B}}^{(n)} \psi_{\mathrm{B}}^{(n')\dagger} \end{aligned}$$

$\mathrm{spec}\,(CC^{\dagger}) = \mathrm{spec}\,(C^{\dagger}C) \pm \{0\} \quad \Rightarrow \quad S_{\mathrm{A}} = S_{\mathrm{B}}$

True for any $\mathcal{M}$ and $\mathcal{N}$.

Schmidt decomposition theorem

For any pure state ψ_{AB} there exist orthonormal systems $\{\psi_{\mathrm{A}}^{(n)}\}$ and $\{\psi_{\mathrm{B}}^{(n)}\}$ such that

$$\psi_{\mathrm{AB}} = \sum_{n} \lambda_n^{\frac{1}{2}} \psi_{\mathrm{A}}^{(n)} \psi_{\mathrm{B}}^{(n)} ,$$

where $\lambda_n^{\frac{1}{2}} > 0$ and $\sum_n \lambda_n = 1$.

- simple consequence of the singular value decomposition theorem
- dimensionalities of $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$ might be different; range of sum determined by the smaller Hilbert space

Schmidt decomposition theorem

$$\psi_{\mathrm{AB}} = \sum_n \lambda_n^{\frac{1}{2}} \psi_{\mathrm{A}}^{(n)} \psi_{\mathrm{B}}^{(n)}$$

$$\rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}\, \psi_{\mathrm{AB}} \psi_{\mathrm{AB}}^{\dagger} = \sum_n \lambda_n \psi_{\mathrm{A}}^{(n)} \psi_{\mathrm{A}}^{(n)\dagger}$$

$$\rho_{\mathrm{B}} = \mathrm{tr}_{\mathrm{A}}\, \psi_{\mathrm{AB}} \psi_{\mathrm{AB}}^{\dagger} = \sum_n \lambda_n \psi_{\mathrm{B}}^{(n)} \psi_{\mathrm{B}}^{(n)\dagger}$$

$$S_{\mathrm{A}} = S_{\mathrm{B}}$$

- ρ_A and ρ_B have identical non-zero eigenvalues λ_n.
- Same entropy from tracing over interior region B as from tracing over exterior region A even if one of $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$ is vastly larger than the other.

Purification

Reverse procedure:

Given ρ_A, find pure ψ_{AB} such that $\rho_A = \mathrm{tr}_B \psi_{AB} \psi_{AB}^\dagger$.

Purification (non-unique but always possible):

- let $\rho_A = \sum_n \lambda_n \psi_A^{(n)} \psi_A^{(n)\dagger}$
- make a fictitious system B (QFT modes, qubits, etc.)
- replicate $\{\psi_B^{(n)}\} = \{\psi_A^{(n)}\}$ ($\mathcal{M} = \mathcal{N}$ for simplicity)
- use $\psi_{AB} = \sum_n \lambda_n^{\frac{1}{2}} \psi_A^{(n)} \psi_B^{(n)}$
- non-uniqueness due to unitary transformations of $\psi_B^{(n)}$
- highly correlated state

Maximally entangled state

Erroneous argument :

$S_{\mathrm{A}} = S_{\mathrm{B}}$, so entropy depends only on common AB boundary, yielding $S = f(\mathrm{Area}) \sim$ Area.

Counterexample:

$\max_{\{\lambda_n\}} S$ with $\sum_n \lambda_n = 1$ gives $\lambda_n = \mathcal{N}^{-1}$, $\max S = \ln \mathcal{N} \sim V$

This entropy is either for A or B subsystem.

$\rho_{\mathrm{A}} = \mathcal{N}^{-1} \sum_n \psi_{\mathrm{A}}^{(n)} \psi_{\mathrm{A}}^{(n)}$

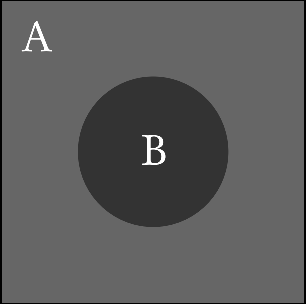
Use purification to get $\psi_{\mathrm{AB}} = \mathcal{N}^{-1/2} \sum_n \psi_{\mathrm{A}}^{(n)} \psi_{\mathrm{B}}^{(n)}$.

Paradox?

Take the maximal entropy state.

Spread exterior states $\psi_{\mathrm{A}}^{(n)}$ (qubits) thinly around the universe.

(Keep correlations, eliminate interactions.)



Let B collapse to a black hole.

Usual semi-classical Hawking calculation: $S_{\mathrm{BH}} \sim A$.

But, $S_{\mathrm{BH}} = S_{\mathrm{E}} + S'$, where $S' > 0$ and $S_{\mathrm{E}} \sim V_{\mathrm{B}}$.

Resolution

Gravitational collapse: not all states can be used in our construction of $\psi_{\rm AB}$.

Different from limiting the actual size of $\mathcal{H}_{\rm B}$ (holography).

Require $\psi_{\rm AB}$ have not already collapsed to a black hole larger than V: $E_{\rm AB}^{(n)} \lesssim R$ for all states n used in the construction.

Use 't Hooft result ($\mathcal{N}_{\rm max} \sim e^{A^{3/4}}$):

$$\lambda_n = \begin{cases} \mathcal{N}_{\rm max}^{-1}, & n \leq \mathcal{N}_{\rm max} \\ \lambda_n = 0, & n > \mathcal{N}_{\rm max} \end{cases}$$

Conclude: $S \sim A^{3/4}$

Collapse and entanglement entropy

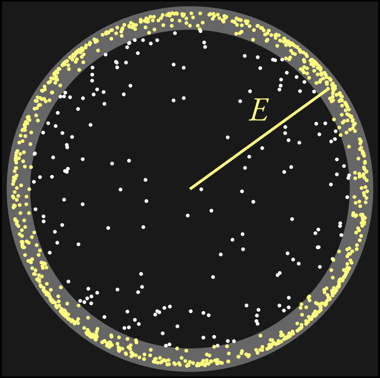
Before:

- begin with states of maximal entropy and impose no-gravitational-collapse condition
- equal probabilities for $n \leq N_{\max}$ states
- $S \sim A^{3/4}$

Now:

- maximize the entropy subject to the collapse condition
- the more energetic state, the lower its probability
- $S \lesssim A^{3/4}$

Density of states



$$\max_{E \leq \text{const}} S \approx \max_{E = \text{const}} S$$

Entropy is maximal when the total energy of the system is maximal.

Maximizing entropy

Maximize: $S = -\sum_n \lambda_n \ln \lambda_n$

Constraints: $\sum_n \lambda_n = 1$ and $\sum_n \lambda_n \epsilon_n = A$, $(\epsilon_n = E_{\mathrm{B}}^{(n)} R)$

Use the method of Lagrange multipliers:

$$\tilde{S} = -\sum_n \lambda_n \ln \lambda_n + \alpha(\sum_n \lambda_n - 1) - \beta(\sum_n \lambda_n \epsilon_n - A).$$

Maximizing entropy

- maximization gives $\lambda_n = Z^{-1}e^{-\beta\epsilon_n}$
  extremal system is thermal
  β is determined by the average total energy condition
  $\langle E\rangle = R$
- to evaluate partition function $Z(\beta) = \sum_n e^{-\beta\epsilon_n}$,
  take continuous limit $Z(\beta) = \int d\epsilon\, \nu(\epsilon)e^{-\beta\epsilon}$
  and use the saddle point method: $\ln Z(\beta) \approx \ln \nu(\epsilon_*) - \beta\epsilon_*$
  the saddle point: $\epsilon = \epsilon_*(\beta)$, a solution of $d\ln\nu(\epsilon)/d\epsilon = \beta$
- find: $\max S \approx \ln \nu(\epsilon)|_{\epsilon=A}$
  use 't Hooft result $\epsilon = ER \sim S_{\text{th}}^{4/3} \sim (\ln \nu)^{4/3}$
  to get $\max S \approx A^{3/4}$
- the result inevitably agrees with the 't Hooft's calculation described earlier
- in d spacetime dimensions: $\max S \approx A^{(d-1)/d}$

Conclusions

The holographic conjecture makes a strong assertion that states with $\langle H \rangle \gtrsim R$ simply do not exist in the Hilbert space.

But why does the universe appear to have d spacetime dimensions if the Hilbert space is only that of a $d-1$ dimensional system?

What happens to unitarity, locality, etc. ?

Conclusions

We suggest an alternative interpretation of black hole entropy bounds.

The no-gravitational-collapse condition on ψ_{AB} places an upper bound on the entanglement (or von Neumann) entropy that can be realized from a region B without forming a black hole larger than B itself.

Highly energetic states remain in the theory, but cannot be used to increase the entropy beyond the area of B in Planck units.

Entropy bounds reflect the limitations that gravity imposes on the construction of pure states or density matrices, but do not require a truncation of the Hilbert space itself.